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OPERATIONAL CALCULI FOR THE EULER OPERATOR

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Dedicated to Professor Megumi Saigo,
on the occasion of his 70th anniversary

Abstract

A direct algebraic construction of a family of operational calculi for the Euler differential operator $\delta = t\frac{d}{dt}$ is proposed. It extends the Mikusiński's approach to the Heaviside operational calculus for the case when the classical Duhamel convolution is replaced by the convolution

$$(f * g)(t) = \Phi_\tau \Big\{ \int_\tau^t f\Big(\frac{t}{\tau}\Big) g(\tau) \frac{d\tau}{\tau} \Big\}$$

in $\mathcal{C}(\triangle)$, where $\triangle$ is a subinterval of $\mathbb{R}_+ = (0, \infty)$, and Φ is a nonzero linear functional on $\mathcal{C}(\triangle)$. An essential difference compared with the Mikusiński's approach is the necessity to cope with the abundance of divisors of zero of the convolutions we are considering. The basic elements of the corresponding operational calculi are exhibited in explicit form. They allow effective solution of nonlocal boundary value problems of the form

$$P(\delta)y = f, \quad \Phi\{\delta^k y\} = \gamma_k, \quad k = 0, 1, 2, ..., \deg P - 1$$

for Euler differential equations.

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1. Introduction

There are not too much attempts to develop operational calculus for the Euler differential operator $\delta = t\frac{d}{dt}$. One of the first could be seen in Bellert [1]. It is a pre-Mikusiński type operational calculus for the "initial" right inverse operator $Lf(t) = \int_1^t f(\tau)\frac{d\tau}{\tau}$ of the Euler operator. In fact, it is only an illustrative example. A Mikusiński-type approach to such an operational calculus is proposed in Gesztelyi [2], where the emphasis is not on the systematic development of the techniques of the corresponding operational calculus, but on applications in number theory. Some hints about such an operational calculus may by traced in Elizarraraz and Verde-Star [3]. As for boundary value problems, connected with the Euler differential equations, we could mention only the paper by Butzer and Jansche [4], where a transform approach, based on the finite Meller transform is proposed.

Our aim here is to embrace the whole variety of the possible operational calculi for the Euler operator for boundary value problems of Taylor type. In this we will follow the approach of Dimovski [5] for the differentiation operator. We believe that the developments connected with the Euler operator present some interest on their own right, due to their transparence and simplicity.

2. Taylor type boundary value problems for Euler differential equations

In order to describe the class of boundary value problems for an Euler differential equation

$$(a_o\delta^n + a_1\delta^{n-1} + \cdots + a_{n-1}\delta + a_n)y = f \tag{1}$$

with $a_j \in \mathbb{C}$, $j = 0, 1, \cdots, n$, we consider it in $\mathcal{C}^n(\mathbb{R}_+)$ where $\mathbb{R}_+ = (0, \infty)$, or in $\mathcal{C}^n(\triangle)$, where $\triangle$ is a subinterval of $\mathbb{R}_+$. For an unification, we assume that $1 \in \triangle$. For the right-hand side function we assume $f \in \mathcal{C}(\mathbb{R}_+)$ or $f \in \mathcal{C}(\triangle)$.

Let Φ be a given linear functional in $\mathcal{C}(\mathbb{R}_+)$ or $\mathcal{C}(\triangle)$. Then, as it is well known, according to Riesz-Markov theorem the functional Φ has a representation of the form

$$\Phi\{f\} = \int_\alpha^\beta f(\tau)d\mu(\tau), \tag{2}$$

with $0 < \alpha \leq \beta < +\infty$ and with a Radon measure $\mu(t)$.

DEFINITION 1. *Taylor-type boundary value conditions* for equation (1), determined by the linear functional Φ are said to be the equations

$$\Phi\{\delta^k y\} = \gamma_k, \quad k = 0, 1, 2, \cdots, n, \tag{3}$$

where γ_k are given real or complex numbers.

We name these boundary value conditions of Taylor-type due to the generalized Taylor formula for the Euler operator, which is a special case of the generalization of the classical Taylor formula, due to Bittner [6] and Przeworska-Rolewicz [7]. We need here only the special case of it for the Euler operator.

In order to state the generalized Taylor formula for the Euler operator, we are to assume additionally that $\Phi\{1\} \neq 0$. Then, without any loss of generality, we may assume

$$\Phi\{1\} = 1. \tag{4}$$

DEFINITION 2. Denote by L the *right inverse operator* of $\delta = t\frac{d}{dt}$, determined as the solution of the elementary boundary value problem

$$\delta(Lf) = f, \quad \Phi\{Lf\} = 0.$$

Obviously,

$$Lf(t) = \int_1^t f(\tau)\frac{d\tau}{\tau} - \Phi_\sigma\Big\{\int_1^\sigma f(\tau)\frac{d\tau}{\tau}\Big\}. \tag{5}$$

THEOREM 1. *Let $g \in \mathcal{C}^n(\mathbb{R}_+)$ (or $g \in \mathcal{C}^n(\triangle)$). Then the following identity holds:*

$$g(t) = \sum_{k=0}^{n-1} \Phi\{\delta^k g\} A_k(t) + L^n(\delta^n g), \tag{6}$$

where $A_k(t) = L^k\{1\}$, $k = 0, 1, 2, \cdots, n-1$, are polynomials.

P r o o f. The proof is elementary. In fact, (6) can be written as the operator identity

$$I = \sum_{k=0}^{n-1}\Big(L^k\delta^k - L^{k+1}\delta^{k+1}\Big) + L^n\delta^n,$$

where I stands for the identity operator in $\mathcal{C}^n$. ∎

For our next considerations we will need the solution of the boundary value problem

$$\delta y - \lambda y = f, \quad \Phi\{y\} = 0, \tag{7}$$

where we do not suppose necessarily $\Phi\{1\} \neq 0$.

The solution of (7) will be denoted by $y = L_\lambda f$, and it has the explicit form

$$L_\lambda f(t) = \int_1^t \Big(\frac{t}{\tau}\Big)^\lambda f(\tau)\frac{d\tau}{\tau} - \frac{t^\lambda}{E(\lambda)}\Phi_\sigma\Big\{\int_1^\sigma \Big(\frac{\sigma}{\tau}\Big)^\lambda f(\tau)\frac{d\tau}{\tau}\Big\}, \tag{8}$$

with $E(\lambda) = \Phi_\tau\{\tau^\lambda\}$. The operator L_λ is said to be *the resolvent operator of the Euler operator* $\delta = t\frac{d}{dt}$ for the boundary value functional Φ.

Our next step should be to find a convolution for L_λ, i.e., a bilinear, commutative and associative operation $f*g$ in $\mathcal{C}(\mathbb{R}_+)$ or $\mathcal{C}(\triangle)$ such that L_λ to be a convolutional operator. Such an operation is exhibited in Dimovski and Hristov [8].

THEOREM 2. *The operation*

$$(f*g)(t) = \Phi_\tau\Big\{\int_\tau^t f\Big(\frac{t}{\sigma}\Big)g(\sigma)\frac{d\sigma}{\sigma}\Big\} \tag{9}$$

is a bilinear, commutative and associative operation in $\mathcal{C}(\triangle)$, *such that*

$$L_\lambda f = \Big\{\frac{t^\lambda}{E(\lambda)}\Big\} * f. \tag{10}$$

P r o o f. The bilinearity and the commutativity of (9) are almost evident, and only the associativity needs a proof. First, let us take

$$f(t) = t^\mu \quad \text{and} \quad g(t) = t^\nu.$$

Then we obtain easily

$$\begin{aligned}\{t^\mu\} * \{t^\nu\} &= \Phi_\tau\Big\{\int_\tau^t \frac{(t\tau)^\mu}{\sigma^\mu}\sigma^\nu\frac{d\sigma}{\sigma}\Big\}\\ &= \frac{E(\mu)t^\nu - E(\nu)t^\mu}{\nu-\mu},\end{aligned}$$

where we use the denotation $E(\lambda) = \Phi_\tau\{\tau^\lambda\}$. The function $E(\lambda)$ is said to be the *Euler indicatrix* of the functional Φ.

Then it is easy to verify that

$$(\{t^\mu\} * \{t^\nu\}) * \{t^\kappa\} = \{t^\mu\} * (\{t^\nu\} * \{t^\kappa\}),$$

provided $\mu \neq \nu \neq \kappa \neq \mu$. Looking on μ, ν and κ as on variables, we differentiate with respect to μ, ν and κ, respectively m, n and k times, and obtain

$$\Big(\{t^\mu(\ln t)^m\} * \{t^\nu(\ln t)^n\}\Big) * \{t^\kappa(\ln t)^k\} = \{t^\mu(\ln t)^m\} * \Big(\{t^\nu(\ln t)^n\} * \{t^\kappa(\ln t)^k\}\Big).$$

Then, passing to the limits $\mu \to +0$, $\nu \to +0$ and $\kappa \to +0$, we obtain

$$\Big(\{(\ln t)^m\} * \{(\ln t)^n\}\Big) * \{(\ln t)^k\} = \{(\ln t)^m\} * \Big(\{(\ln t)^n\} * \{(\ln t)^k\}\Big)$$

for arbitrary $m, n, k \in \mathbb{N}_0$. From the bilinearity of (9), we may assert that

$$\Big(\{P(\ln t)\} * \{Q(\ln t)\}\Big) * \{R(\ln t)\} = \{P(\ln t)\} * \Big(\{Q(\ln t)\} * \{R(\ln t)\}\Big)$$

for arbitrary polynomials P, Q and R. If $f, g, h \in \mathcal{C}(\triangle)$, then they can be approximated almost uniformly by function of the form $P(\ln t)$, $Q(\ln t)$ and $R(\ln t)$ respectively (according to Weierstrass' approximation theorem). Thus the associativity relation

$$(f * g) * h = f * (g * h)$$

is proven for arbitrary $f, g, h \in \mathcal{C}(\triangle)$.

The proof of (10) is a matter of a direct check. ∎

3. Mikusiński-type operational calculi for the Euler operator

Further, we assume again that $\Phi\{1\} = 1$. As a special case of (10), we have

$$Lf = \{1\} * f \tag{11}$$

and we may extend the Mikusiński's approach to the operator L.

Considering $\mathcal{C}(\triangle)$ as a commutative and associative algebra with the multiplication operation (9), we are faced with the fact that in the algebra $\{\mathcal{C}(\triangle), *\}$ there may be a plenty of divisors of zero. In Dimovski [5] it is shown how to proceed in such a case.

Let us denote by $\mathcal{C}$ the space $\mathcal{C}(\mathbb{R}_+)$ or $\mathcal{C}(\triangle)$. Denote by $\mathcal{D}$ the subset of $\mathcal{C}$ consisting of all nonzero nondivisors of zero of the convolution algebra $(\mathcal{C}, *)$. The set $\mathcal{D}$ is nonempty, since at least $\{t^\lambda\} \in \mathcal{D}$ for $E(\lambda) \neq 0$.

Indeed, assume that $\{t^\lambda\} * f = 0$ for $f \in \mathcal{C}$. By (10), we have $L_\lambda f = 0$. Since L_λ is a right inverse operator of $\delta - \lambda$, then $(\delta - \lambda)L_\lambda f = f = 0$. This means that $\{t^\lambda\} \in \mathcal{D}$ for $E(\lambda) \neq 0$. It is possible a complete characterization of $\mathcal{D}$ in the case of a compact interval $\triangle$, but this is not essential for our approach. Further we follow the scheme, described in Dimovski [5] and [10].

Consider the set $\mathcal{C} \times \mathcal{D}$ and define the following equivalence relation in it:

$$(f, g) \sim (f_1, g_1) \Longleftrightarrow f * g_1 = g * f_1. \tag{12}$$

The convolution fraction $\frac{f}{g}$ for $f \in \mathcal{C}$ and $g \in \mathcal{D}$ is defined as the equivalence class $[(f, g)]$ of (12) containing the couple (f, g). It remains to define the operations addition and multiplication of convolution fractions:

$$\frac{f}{g} + \frac{h}{k} = \frac{f * k + g * f}{g * k}, \tag{13}$$

and

$$\frac{f}{g} \cdot \frac{h}{k} = \frac{f * h}{g * k}. \tag{14}$$

Thus the set M of the convolution fractions becomes a commutative ring. Following the same procedure as in Mikusiński [9] we embed both the convolution algebra $\mathcal{C}$ and the number field $\mathbb{C}$ into M by the maps

$$\chi(f) = \frac{f * g}{g}, \quad g \in \mathcal{D} \tag{15}$$

and

$$\eta(\alpha) = \frac{\alpha f}{f}, \quad f \in \mathcal{D}. \tag{16}$$

The operator L, according to (11), coincides with the convolution operator $\{1\}*$, but following the Mikusiński's approach [9], we identify it with the constant function $f(t) \equiv 1$ for $t \in \triangle$, i.e., $L = \{1\}$. As in Mikusiński's calculus, it is different from the unit element 1 of the ring of the convolution fractions M. The reciprocal element L^{-1} of L plays a basic role in our operational calculus, and we denote it by S, i.e.,

$$S = L^{-1}. \tag{17}$$

The element S of M may be called the *algebraic Euler operator*. It is applicable to each $f \in \mathcal{C}$ in the sense of the multiplication in M, i.e.,

$Sf = S \cdot f$ for $f \in \mathcal{C}$. If $f \in \mathcal{C}^1$, then to f it is applicable the ordinary Euler operator $\delta = t\frac{d}{dt}$:

$$\delta f = \{tf'(t)\}.$$

It may do not coincide with Sf. Then the exact relationship between δf and Sf is given by the following formula, which could be called the *basic formula of the operational calculus for the Euler operator.*

THEOREM 3. *If $f \in \mathcal{C}^1(\triangle)$, then*

$$\delta f = Sf - \Phi\{f\}, \tag{18}$$

where $\Phi(f)$ is the "numerical operator" $\frac{\{\Phi\{f\}\}}{\{1\}}$.

P r o o f. We have

$$L(\delta f) = f(t) - f(0) - \Phi_\tau\{f(\tau) - f(0)\} = f(t) - \Phi\{f\}.$$

This identity can be written as

$$L(\delta f) = f - \Phi\{f\}L$$

in the ring M. It remains to multiply both sides of it by $S = L^{-1}$:

$$\delta f = Sf - \Phi\{f\}.$$

■

COROLLARY 1. *If $f \in \mathcal{C}^n(\triangle)$, then*

$$\delta^n f = S^n f - \Phi\{f\}S^{n-1} - \Phi\{\delta f\}S^{n-2} - \cdots - \Phi\{\delta^{n-1} f\}. \tag{19}$$

This identity can be obtained from (18) by induction. Let us remark that (19) in fact coincides with the Taylor formula (6) for the Euler operator L.

THEOREM 4. *Let $\lambda \in \mathbb{C}$ be such that $E(\lambda) \neq 0$. Then*

$$\frac{1}{S-\lambda} = \left\{\frac{t^\lambda}{E(\lambda)}\right\} \tag{20}$$

and

$$\frac{1}{(S-\lambda)^k} = \left\{\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\Big(\frac{t^\lambda}{E(\lambda)}\Big)\right\}, \quad \textit{for} \quad k > 1. \tag{21}$$

P r o o f. In fact, (20) is only an alternative form of (10). Indeed, $L_\lambda f = y$ is the solution of the equation $\delta y - \lambda y = f$ with boundary value condition $\Phi\{y\} = 0$. Using formula (18), we obtain $Sy - \lambda y = f$, or $(S-\lambda)y = f$. Under the assumption $E(\lambda) \neq 0$, $S - \lambda$ is not a divisor of zero. Hence

$$y = \frac{1}{S-\lambda} f = L_\lambda f = \left\{ \frac{t^\lambda}{E(\lambda)} \right\} * f\,.$$

Therefore

$$\frac{1}{S-\lambda} = L_\lambda = \left\{ \frac{t^\lambda}{E(\lambda)} \right\}.$$

As for (21), it can be proven by induction. Formally it could be obtained from (20) by differentiation with respect to λ. ■

4. Applications for solution of Taylor-type boundary value problems for Euler differential equations

Let us consider an Euler differential equation

$$(a_0\delta^n + a_1\delta^{n-1} + \cdots + a_{n-1}\delta + a_n)y = f$$

in an interval $\triangle \subset \mathbb{R}_+$ with $1 \in \triangle$. For the brevity sake, we will write it as

$$P(\delta)y = f,$$

where

$$P(\mu) = a_0\mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu + a_n.$$

The coefficients of the polynomial $P(\mu)$ are, in general, complex numbers. We assume also that $a_0 \neq 0$, i.e., that P is exactly of degree n. For the right-hand side f we assume that $f \in \mathcal{C}(\triangle)$.

Further, we take an arbitrary nonzero linear functional Φ in $\mathcal{C}(\triangle)$. It has the form (2).

We consider the boundary value problem

$$P(\delta)y = f, \quad \Phi\{\delta^k y\} = \gamma_k, \quad k = 0, 1, 2, \cdots, n-1,\ n = \deg P. \tag{22}$$

Considering the solution y as an element of the ring of the convolutional fractions M, boundary value problem (22) is equivalent to a single equation. This algebraization of the boundary value problem is done analogously to the procedure, used in Mikusiński [9]. One should use the basic formula (19) and to substitute

$$\delta^k y = S^k y - \gamma_0 S^{k-1} - \gamma_1 S^{k-2} - \cdots - \gamma_{k-2} S - \gamma_{k-1}, \quad k = 0, 1, 2, \cdots, n-1$$

in the equation $P(\delta)y = f$. Thus we obtain

$$P(S)y = f + Q(S), \tag{23}$$

where

$$P(S) = a_0 S^n + \cdots + a_{n-1} S + a_n$$

and

$$Q(S) = \sum_{k=0}^{n} \sum_{l=0}^{k-1} a_k \gamma_l S^{k-l-1}.$$

The formal solution of (23) would be

$$y = \frac{1}{P(S)} f + \frac{Q(S)}{P(S)},$$

provided $P(S)$ is not a divisor of zero in M. The problem to decide weather $P(S)$ is a divisor of zero or not is an easy one. Its solution is given by the following theorem.

THEOREM 5. *The element*

$$P(S) = a_0 S^n + a_1 S^{n-1} + \cdots + a_{n-1} S + a_n$$

is a divisor of zero in M if and only if a root of $P(\mu)$ is a root of $E(\lambda)$ too.

P r o o f. Let $P(\mu) = (\mu - \mu_k) P_1(\mu)$ and let $E(\mu_k) = 0$. This means that μ_k is an eigenvalue of the problem (7), i.e., that there exists a nonzero function y such that $\delta y - \mu_k y = 0$ and $\Phi\{y\} = 0$. Thus $(S - \lambda)y = 0$, and hence $S - \lambda$ is a divisor of zero. The converse assertion is easier to prove. ■

After the characterization of the divisors of zero of the form $P(S)$, we may proceed to practical solution of boundary value problems of the form (22).

EXAMPLE 1. Consider the Euler differential equation

$$t^2 \frac{dy}{dt} + \alpha t \frac{dy}{dt} + \beta y = f(t)$$

with boundary value conditions

$$\int_1^2 y(\tau)d\tau = \gamma_o \quad \text{and} \quad \int_1^2 \tau y'(\tau)d\tau = \gamma_1 .$$

Here the functional is $\Phi\{f\} = \int_1^2 f(\tau)d\tau$, and the normalizing condition $\Phi\{1\} = 1$ is fulfilled. Expressing $t^2\frac{d^2}{dt^2}$ as

$$\delta^2 - \delta,$$

we obtain the equation in the form

$$[\delta^2 + (\alpha - 1)\delta + \beta]y = f(t).$$

Denoting $P(\delta) = \delta^2 + (\alpha - 1)\delta + \beta$, we should distinguish 3 cases: a) $(\alpha - 1)^2 - 4\beta > 0$, b) $(\alpha - 1)^2 - 4\beta = 0$, and c) $(\alpha - 1)^2 - 4\beta < 0$.

In the case a)

$$\mu_{1,2} = \frac{1}{2}\Big[1 - \alpha \pm \sqrt{(\alpha - 1)^2 - 4\beta}\Big]$$

are the roots of $P(\mu)$; in the case b) we have the double root

$$\mu_1 = \mu_2 = \frac{1}{2}(1 - \alpha),$$

and in the case c) we have the complex roots

$$\mu_{1,2} = \frac{1}{2}\Big[(1 - \alpha) \pm i\sqrt{4\beta - (\alpha - 1)^2}\Big].$$

The Euler indicatrix of the functional Φ is

$$E(\lambda) = \frac{2^{\lambda+1} - 1}{\lambda + 1}$$

and its roots are

$$\lambda_n = 2ni\pi \ln 2 - 1, \quad n \in \mathbb{Z} \setminus \{0\}.$$

Let us consider the case a). Then the problem we are considering reduces to the single equation

$$P(S)y = f + \gamma_0(S + \alpha - 1) + \gamma_1.$$

Its formal solution is

$$y = \frac{1}{(S-\mu_1)(S-\mu_2)} f + \frac{\gamma_0(S+\alpha-1)+\gamma_1}{(S-\mu_1)(S-\mu_2)}.$$

The roots μ_1 and μ_2 are different from $\lambda_n = 2ni\pi \ln 2 - 1$, $n \in \mathbb{Z} \setminus \{0\}$ and hence $P(S)$ is not a divisor of zero. We have

$$\frac{1}{(S-\mu_1)(S-\mu_2)} = \frac{1}{\sqrt{(\alpha-1)^2 - 4\beta}} \Big(\frac{1}{S-\mu_2} - \frac{1}{S-\mu_1} \Big)$$

and

$$\frac{\gamma_0(S+\alpha-1)+\gamma_1}{(S-\mu_1)(S-\mu_2)}$$
$$= \frac{1}{\sqrt{(\alpha-1)^2-4\beta}} \Big(\frac{\gamma_0(\mu_1+\alpha-1)+\gamma_1}{S-\mu_1} - \frac{\gamma_0(\mu_2+\alpha-1)+\gamma_1}{S-\mu_2} \Big).$$

Hence the solution, we are looking for is

$$y = \frac{1}{\sqrt{(\alpha-1)^2+\beta^2}} \Big[\frac{t^{\mu_2}}{E(\mu_2)} - \frac{t^{\mu_1}}{E(\mu_1)} \Big] * f + At^{\mu_1} + Bt^{\mu_2}$$

with

$$A = \frac{\gamma_0(\mu_1+\alpha-1)+\gamma_1}{\sqrt{(\alpha-1)^2-4\beta}} \quad \text{and} \quad B = -\frac{\gamma_0(\mu_2+\alpha-1)}{\sqrt{(\alpha-1)^2-4\beta}}.$$

A special interest presents the case when $\gamma_k = 0$, $k = 0, 1, 2, \cdots, n-1$.

THEOREM 6. *If g is the solution of the problem*

$$P(\delta)g = 1, \quad \Phi\{\delta^k g\} = 0, \quad k = 0, 1, 2, \cdots, n-1, \tag{24}$$

then a solution of the problem

$$P(\delta)y = f, \quad \Phi\{\delta^k y\} = 0, \quad k = 0, 1, 2, \cdots, n-1 \tag{25}$$

has the form

$$y = \delta(g * f) = t\frac{d}{dt}\Phi_\tau\Big\{ \int_\tau^t g\Big(\frac{t}{\tau}\Big) f(\tau)\frac{d\tau}{\tau} \Big\}. \tag{26}$$

P r o o f. We try to verify directly that (26) is a solution of (25). To this end, we use the formula

$$\delta(g * f) = (\delta g) * f - \Phi(g) f$$

valid for $g \in \mathcal{C}^1(\triangle)$ and $f \in \mathcal{C}(\triangle)$. Thus we obtain

$$P(\delta)y = \delta(P(\delta)g * f) = \delta(\{1\} * f) = f,$$

where we use the boundary value conditions $\Phi(\delta^k g) = 0, \quad k = 0, 1, 2, \cdots, n-1$. Hence $y = \delta(g * f)$ is a solution of the Euler differential equation

$P(\delta)y = f$. In order to prove that $y = \delta(g * f)$ satisfies the boundary value conditions $\Phi(\delta^k y) = 0$, $k = 0, 1, 2, \cdots, n-1$, we use the general property of the convolution (9),

$$\Phi\{p * q\} = 0$$

for arbitrary $p, q \in \mathcal{C}(\triangle)$ (see Dimovski and Hristov [8]). ∎

Theorem 6 extends the Duhamel principle to the Euler operator.

References

[1] S. Bellert, *On the continuation of the idea of Heaviside in the operational calculus. J. Franklin Inst.* **276**, No 5(1963), 411-440.

[2] E. Gesztelyi, The application of the operational calculus in the theory of numbers. *Colloq. Math. Soc. Janos Bolyai* **2** 1968), 51-104.

[3] D. Elizarraraz and L. Verde-Star, On a class of differential equations that contains the equations of Euler and Chebyshev. *Adv. Appl. Math.* **19** (1997), 514-528.

[4] P. L. Butzer and S. Jansche, The finite Mellin transform, Mellin-Fourier series, and the Mellin-Poisson summation formula. *Rendiconti del Circolo Matematico di Palermo. Seria II, Supl.* **52** (1998), 55-81.

[5] I. H. Dimovski, Nonlocal operational calculi. *Proc. of Steklov Ins. Math. Inst.* **3** (1995), 53-65.

[6] R. Bittner, Operational calculus in linear spaces. *Studia Math.* **21** (1962), 1-18.

[7] D. Przeworska-Rolewicz, Algebraic theory of right invertible operators, *Studia Math.* **48** (1973), 129-144.

[8] I. H. Dimovski and V. Z. Hristov, Commutants of the Euler operator. *C. R. Acad. Bulg. Sci.*, **59**, No 1(2006), 125-129.

[9] J. Mikusiński, *Operational Calculus.* Pergamon Press, Oxford (1959).

[10] I. H. Dimovski, *Convolution Calculus.* Kluwer, Dordrecht (1990).

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